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Over-reflection of internal–inertial waves from the mixed layer

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Near-inertial oscillations associated with downward energy propagation are commonly observed in the upper ocean. Stern (1977) has suggested that these observations may be internal–inertial waves over-reflected from the shear zone at the base of the mixed layer. In this paper we develop a criterion for over-reflection as a function of wavenumber and frequency for a class of shear flows in the mixed layer. By examining the vertical profile of the vertical wave action flux we demonstrate that the source of the over-reflection is the shear at the base of the mixed layer, which is maintained by the wind-induced turbulent Reynolds stress, here parametrized as a body force. The relationship between over-reflection and the wave-induced Lagrangian-mean flow is determined. We also determine a criterion for unstable waves, and show that these are contiguous in wavenumber–frequency space with points of resonant over-reflection. However, the growth rates of these unstable waves are quite small, and in practice unstable waves will be indistinguishable from waves generated by over-reflection.

1. Introduction

Near-inertial oscillations are a prominent feature of upper-ocean observations. Often these observations also show a dominant upward phase propagation implying downward energy propagation (see e.g. Leaman & Sanford 1975; Rossby & Sanford 1976; Fu 1981). Thus there is evidence for the generation of internal–inertial waves by the action of the wind stress on the ocean surface. Pollard (1970) has developed a model in which the wind stress is parametrized as a body force acting throughout the mixed layer, and has demonstrated that wind stress transience can generate near-inertial oscillations in the upper ocean with amplitudes consistent with observations (Pollard & Millard 1970). Kroll (1975) proposed an alternative model in which the forcing was due to the Ekman suction velocity arising from a divergence in the Ekman transport. Both these models produce internal–inertial waves whose horizontal scale is that of the imposed wind stress.

However, Stern (1977) has pointed out that observed internal–inertial waves often have horizontal scales much smaller than the gross dimensions of atmospheric forcing. Hence he proposed an alternative model in which the wind stress acts to produce a shear flow within the mixed layer. Internal–inertial waves incident on this shear layer from below can then be over-reflected in some circumstances. For certain values of the wave frequency and horizontal wavenumber the reflection coefficient was found to be infinite corresponding to resonant over-reflection. The preferred horizontal scale for these waves is $(NH)^2 (fU)^{-1}$, where N is the Brunt–Väisälä frequency in the region below the mixed layer, H is the mixed-layer depth, f is the Coriolis parameter and U is a typical speed for the shear flow in the mixed layer (i.e. $\tau(\rho_0 fH)^{-1}$, where τ is the wind stress). For typical oceanic parameters this horizontal scale is smaller than the dimensions of atmospheric systems. The corresponding wave frequencies are close to the inertial frequency ($|\sigma/f|-1$ scales with $U(NH)^{-1}$, where σ is the wave frequency). Unlike the models of Pollard (1970) or Kroll (1975), this over-reflection mechanism for the generation of internal–inertial waves does not require the presence of either wind-stress transience or horizontal variability. Instead Stern's model requires only a steady wind stress and the horizontal scale of the waves is determined intrinsically.

In this paper we propose to re-examine and develop Stern's model. In §2 we formulate the linearized equations of motion and determine the criterion for over-reflection for a number of typical mixed-layer shear flows, including one case considered by Stern. One of the interesting features of the over-reflection mechanism considered here is that, unlike most examples of over-reflection found in the literature (Acheson 1976), it does not require the presence of a critical layer. In order to understand this aspect and to determine the mechanism for over-reflection in this present case, we examine the vertical wave action flux. Although this is constant in the region below the mixed layer, it is not constant within the mixed layer itself owing to the presence of the shear flow which is maintained by a body force which in turn is a parametrization of the turbulent Reynolds stress induced by the applied wind stress. By examining the vertical dependence of the vertical wave action flux we show that the source of over-reflection is the shear at the base of the mixed layer.

With the development by Andrews & McIntyre (1978*a*, *b*) of a general theory of wave–mean-flow interaction and wave action, we are in a position to put the wave action considerations of §2 into a more general setting. Hence in §3 we derive the wave action equation for a more general class of waves than that considered in §2, and also describe its relationship to the wave energy equation. Then in §4 we derive the equations for the wave-induced Lagrangian-mean flow, and show how this is related to divergence of wave action flux. In this section we also obtain the total energy equation and its relationship to wave action.

Finally in §5 we return to the formulation of §2 but instead of looking for neutral, over-reflected waves we search for unstable waves. We show that resonantly over-reflected waves are contiguous in frequency–wavenumber space with unstable waves. However, for typical oceanic parameters, we find that these unstable waves have very small growth rates, and in practice will be indistinguishable from waves produced by over-reflection.

2. Linear theory

The coordinate system is described in figure 1. Within the mixed layer ($0 < z < H$) the basic state shear flow $\boldsymbol{u}_0(z)$ has components $u_0(z)$ and $v_0(z)$ in the x- and y-directions respectively. The shear flow is maintained by the stress $\boldsymbol{F}_0(z)$ whose components are $F_0(z)$ and $G_0(z)$, where

$$-\rho_0 f v_0 = \frac{\partial F_0}{\partial z}, \quad \rho_0 f u_0 = \frac{\partial G_0}{\partial z}. \tag{2.1}$$

At the top of the mixed layer ($z = H$) $\boldsymbol{F}_0$ equals the wind stress $\boldsymbol{\tau}$. At the base of the mixed layer ($z = 0$) and throughout the stratified ocean ($z < 0$) both $\boldsymbol{F}_0$ and $\boldsymbol{u}_0$ vanish. It follows that the wind stress is given by the Ekman transport relation

$$\boldsymbol{\tau} = f\boldsymbol{k} \times \int_0^H \rho_0 \boldsymbol{u}_0(z)\, \mathrm{d}z. \tag{2.2}$$

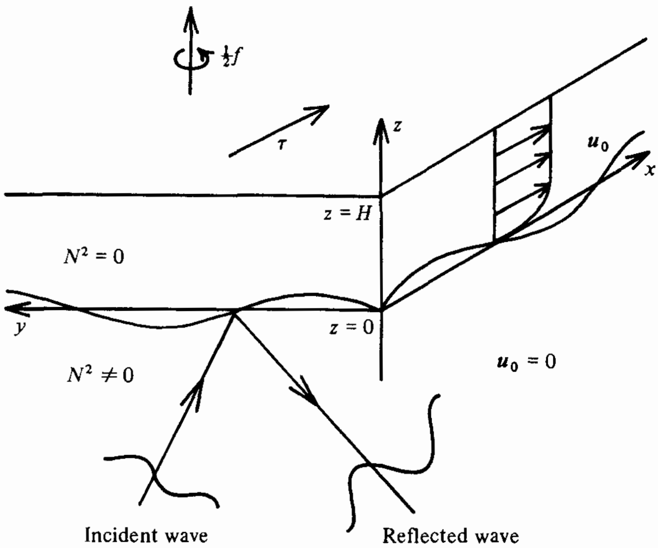


FIGURE 1. The coordinate system.

Here $\boldsymbol{k}$ is a unit vector in the z-direction. In the mixed layer the density ρ_0 is constant, but in the stratified ocean ($z < 0$) we shall assume that the Brunt–Väisälä frequency N is constant, where $\rho_0 N^2 = -g\,\mathrm{d}\rho_0/\mathrm{d}z$. We shall also assume that both ρ_0 and $\boldsymbol{u}_0$ are continuous functions of z, and in particular that they are continuous at $z = 0$.

We shall denote the linearized perturbations to this basic state with a subscript 1. Using the Boussinesq and hydrostatic approximations, the linearized perturbation equations are

$$\frac{\mathrm{D}u_1}{\mathrm{D}t}+w_1\frac{\partial u_0}{\partial z}-fv_1+\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} = 0, \quad \frac{\mathrm{D}v_1}{\mathrm{D}t}+w_1\frac{\partial v_0}{\partial z}+fu_1+\frac{1}{\rho_0}\frac{\partial p_1}{\partial y} = 0, \qquad (2.3a, b)$$

$$g\rho_1+\frac{\partial p_1}{\partial z} = 0, \quad \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{g\rho_1}{\rho_0}\right)-N^2 w_1 = 0, \quad \frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+\frac{\partial w_1}{\partial z} = 0, \qquad (2.3c\text{–}e)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}. \qquad (2.3f)$$

Note that in the mixed layer ($0 < z < H$) ρ_0 is a constant and so N is zero, while in the stratified ocean u_0 and v_0 are zero. We shall find it useful to reformulate these equations in terms of the Lagrangian particle displacements ξ_1, η_1 and ζ_1. In §4 we shall define these precisely using the generalized Lagrangian-mean formulation of Andrews & McIntyre (1978*a*). Here it suffices to note the linearized relations

$$\frac{\mathrm{D}\xi_1}{\mathrm{D}t} = u_1+\zeta_1\frac{\partial u_0}{\partial z}, \quad \frac{\mathrm{D}\eta_1}{\mathrm{D}t} = v_1+\zeta_1\frac{\partial v_0}{\partial z}, \quad \frac{\mathrm{D}\zeta_1}{\mathrm{D}t} = w_1. \qquad (2.4a\text{–}c)$$

It may then be shown that ($2.3a$–f) become

$$\frac{\mathrm{D}^2\xi_1}{\mathrm{D}t^2}-fv_1+\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} = 0, \quad \frac{\mathrm{D}^2\eta_1}{\mathrm{D}t^2}+fu_1+\frac{1}{\rho_0}\frac{\partial p_1}{\partial y} = 0, \qquad (2.5a, b)$$

$$N^2\zeta_1+\frac{1}{\rho_0}\frac{\partial p_1}{\partial z} = 0, \quad \frac{\partial \xi_1}{\partial x}+\frac{\partial \eta_1}{\partial y}+\frac{\partial \zeta_1}{\partial z} = 0. \qquad (2.5c, d)$$

The boundary conditions are that $\zeta_1 = 0$ at $z = H$, and that ζ_1 and p_1 are continuous at $z = 0$. As $z \to -\infty$ we shall specify an incident wave, and require the remaining part of the solution to be outgoing.

Before proceeding with the analysis, we shall comment on the various approximations that have been used here. The Boussinesq approximation involves the neglect of the density fluctuations in (2.3*a*, *b*), and requires that $N^2Hg^{-1} \ll 1$; this condition is readily met in oceanic conditions. The hydrostatic approximation involves the neglect of the vertical acceleration in (2.3*c*), and requires that $\sigma^2 \ll N^2$ where σ is the wave frequency, and also that $|lH| \ll 1$, where l is a horizontal wavenumber. These conditions are verified *a posteriori* as our solutions are for wave frequencies σ close to the inertial frequency f, with $f^2 \ll N^2$, and have a large horizontal wavelength compared with the mixed-layer depth. The hydrostatic approximation also involves the neglect of the horizontal component of the Earth's rotation in (2.3*a*–*c*); this can be justified in midlatitudes for near-inertial frequencies of large horizontal scale. Finally, the rigid-lid condition (i.e. $\zeta_1 = 0$ at $z = H$) is justified when the divergence parameter $f^2(gHl^2)^{-1} \ll 1$, and this we have verified *a posteriori*.

Next we seek solutions of the form

$$\zeta_1 = \hat{\zeta}_1(z)\exp(\mathrm{i}s) + \text{c.c.}, \quad \text{etc.}, \tag{2.6a}$$

where

$$s = kx + ly - \sigma t - \theta. \tag{2.6b}$$

Here θ is a phase-shift parameter introduced so that we can define ensemble averages; for solutions of the form (2.6*a*) these averages are equivalent to time or space averages, but in §§3 and 4 we shall extend the definition of an ensemble average. From (2.6*a*) it follows that

$$-\mathrm{i}\omega(\omega^2 - f^2)\,\hat{\xi}_1 = (\omega k + \mathrm{i}lf)\frac{\hat{p}_1}{\rho_0} + \left(-f^2\frac{\partial u_0}{\partial z} - \mathrm{i}\omega f\frac{\partial v_0}{\partial z}\right)\hat{\zeta}_1, \tag{2.7a}$$

$$-\mathrm{i}\omega(\omega^2 - f^2)\,\hat{\eta}_1 = (\omega l - \mathrm{i}kf)\frac{\hat{p}_1}{\rho_0} + \left(-f^2\frac{\partial v_0}{\partial z} + \mathrm{i}\omega f\frac{\partial u_0}{\partial z}\right)\hat{\zeta}_1, \tag{2.7b}$$

$$(\omega^2 - f^2)\frac{\partial \hat{\zeta}_1}{\partial z} = (k^2 + l^2)\frac{\hat{p}_1}{\rho_0} + \frac{f}{\omega}\left(f\frac{\partial \omega}{\partial z} - \mathrm{i}\omega k\frac{\partial v_0}{\partial z} + \mathrm{i}\omega l\frac{\partial u_0}{\partial z}\right)\hat{\zeta}_1, \tag{2.7c}$$

$$N^2\hat{\zeta}_1 + \frac{1}{\rho_0}\frac{\partial \hat{p}_1}{\partial z} = 0, \tag{2.7d}$$

where

$$\omega = \sigma - ku_0 - lv_0. \tag{2.7e}$$

Equations (2.7*c*, *d*) form a pair of coupled equations for $\hat{\zeta}_1$ and $\hat{p}_1$. The vertical component of wave action flux is

$$B_3 = -2\,\mathrm{Im}\,(\hat{p}_1\,\hat{\zeta}_1^*). \tag{2.8}$$

Using (2.7*c*, *d*), it follows that

$$\frac{\partial B_3}{\partial z} = D, \tag{2.9a}$$

where

$$(\omega^2 - f^2)\,D = \frac{f^2}{\omega}\frac{\partial \omega}{\partial z}B_3 + f\left(l\frac{\partial u_0}{\partial z} - k\frac{\partial v_0}{\partial z}\right)2\,\mathrm{Re}\,(\hat{p}_1\,\hat{\zeta}_1^*). \tag{2.9b}$$

For solutions of the form (2.6*a*), ωB_3 is the vertical wave energy flux. In $z < 0$, D is zero and B_3 is constant; since $\omega = \sigma$ and is also constant, the sign of B_3 determines whether or not the solutions are outgoing. However, in the mixed layer $(0 < z < H)$ D is generally non-zero, and wave action flux is not constant.

In $z < 0$, the solution of (2.7*c*, *d*) is

$$\zeta_1 = I \exp(\mathrm{i}mz) + R \exp(-\mathrm{i}mz), \tag{2.10a}$$

where

$$m^2(\sigma^2 - f^2) = N^2(k^2 + l^2) \quad \text{and} \quad \sigma m < 0. \tag{2.10b, c}$$

Here m is the vertical wavenumber, and its sign has been chosen so that I is the incident-wave amplitude and R is the reflected-wave amplitude; note that the vertical group velocity is $-(\sigma m)^{-1}(\sigma^2 - f^2)$. The vertical wave action flux is

$$B_3 = \frac{2\rho_0 N^2}{m}(-|I|^2 + |R|^2) \quad (z < 0). \tag{2.11}$$

The vertical wave energy flux is σB_3, and the criterion for over-reflection is $\sigma B_3 < 0$ in $z < 0$. Since B_3 vanishes at $z = H$, a necessary condition for over-reflection is that $\sigma B_3 < 0$ somewhere in $0 < z < H$, or that $\sigma D > 0$ somewhere in $0 < z < H$ (see (2.9*a*)).

In the mixed layer, $0 < z < H$, (2.7*d*) shows that $\hat{p}_1$ is constant, and equal to its value at $z = 0$. Integration of (2.7*c*) and application of the boundary condition at $z = H$ then shows that

$$\zeta_1 = \frac{\hat{p}_1}{\rho_0}\frac{(k^2+l^2)(\omega^2 - f^2)^{\frac{1}{2}}}{\omega}\exp(-\mathrm{i}\phi)\int_H^z \frac{\omega(z')\exp(\mathrm{i}\phi(z'))}{\{\omega^2(z') - f^2\}^{\frac{3}{2}}}\,\mathrm{d}z' \quad (0 < z < H), \tag{2.12a}$$

where

$$\phi(z) = \int_0^z \left\{k\frac{\partial v_0}{\partial z'}(z') - l\frac{\partial u_0}{\partial z'}(z')\right\}\{\omega^2(z') - f^2\}^{-1}\,\mathrm{d}z'. \tag{2.12b}$$

For simplicity we shall assume that there are no critical layers, and so $\omega \neq f$ for any value of z. Applying the boundary condition that ζ_1 is continuous at $z = 0$, it follows that

$$\frac{R}{I} = \frac{\gamma + 1}{\gamma - 1}, \tag{2.13a}$$

where

$$\gamma = \frac{\mathrm{i}m(\sigma^2 - f^2)^{\frac{3}{2}}}{\sigma}\int_0^H \frac{\omega \exp(\mathrm{i}\phi)\,\mathrm{d}z}{\{\omega^2 - f^2\}^{\frac{3}{2}}}. \tag{2.13b}$$

Equation (2.13*a*) agrees with the result obtained by Stern (1977), and shows that over-reflection occurs whenever

$$\mathrm{Re}(\gamma) > 0. \tag{2.14}$$

In particular, resonant over-reflection $(I = 0)$, occurs when $\gamma = 1$. The vertical wave action flux is readily evaluated from (2.8) and (2.12*a*); in particular, at $z = 0$,

$$B_3 = \frac{2m|\hat{p}_1|^2}{\rho_0 N^2}\mathrm{Re}(\gamma). \tag{2.15}$$

Since $\sigma B_3 < 0$ for over-reflection and $\sigma m < 0$ (2.10*c*), this confirms (2.14) as the condition for over-reflection. From an examination of (2.12*b*) and (2.13*b*) it is apparent that for (2.14) to be satisfied the wavenumber vector $\boldsymbol{\kappa} = (k, l)$ must have a component normal to the basic shear flow $\boldsymbol{u}_0(z)$. Further, following Stern (1977), if we suppose that the basic shear flow is small, (i.e. $|\boldsymbol{\kappa}\cdot\boldsymbol{u}_0| \ll |f|$), then an expansion of (2.12*b*) and (2.13*b*) shows that

$$\operatorname{Re}(\gamma) \approx \frac{m\boldsymbol{\kappa}\cdot\boldsymbol{\tau}}{\rho_0(\sigma^2-f^2)}, \tag{2.16}$$

where we have used the Ekman transport relation (2.2). Recalling (2.10*c*), it follows that there is over-reflection for $\sigma\boldsymbol{\kappa}\cdot\boldsymbol{\tau} < 0$, i.e. for waves travelling upwind.

To make further progress in analysing (2.13*b*) we at first assume that the waves are normal to the shear flow and set $v_0(z) \equiv 0$ and $k = 0$. Then ω (2.7*e*) is identically equal to a constant, σ, and (2.12*b*) and (2.13*b*) become

$$\phi(z) = -flu_0(z)\,(\sigma^2-f^2)^{-1}, \quad \gamma = \mathrm{i}m\int_0^H \exp\left(\mathrm{i}\phi(z)\,\mathrm{d}z.\right. \tag{2.17a, b}$$

(*a*) *'Slab' flow.* Suppose further that $u_0(z)$ is a constant U_s throughout $0 < z < H$, except in a very thin layer near $z = 0$. This case was considered by Stern (1977), who showed that (2.17*b*) can be approximated by

$$\gamma = \mathrm{i}mH\exp\{-\mathrm{i}flU_\mathrm{s}(\sigma^2-f^2)^{-1}\}. \tag{2.18}$$

Resonant over-reflection occurs for

$$\frac{flU_\mathrm{s}}{\sigma^2-f^2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \tag{2.19a}$$

and

$$mH = \pm 1. \tag{2.19b}$$

In figure 2(*a*) we show a contour plot of $|R/I|$ (2.13*a*) as a function of $|\sigma/f|$ and lH. Note that resonant over-reflection occurs at a decreasing sequence of frequencies slightly greater than the inertial, and a corresponding decreasing set of wavenumbers. However, these points are interspersed with points of resonant absorption where $|R/I| = 0$ and $\gamma = -1$. For this special case D (2.9*b*) is zero throughout the mixed layer, except at $z = 0$, where it has a δ-function singularity; as a consequence B_3 is zero throughout the mixed layer, and then jumps to a non-zero value at $z = 0$. The source of the over-reflection is thus the strong shear at the base of the mixed layer.

(*b*) *Linear shear.* Next suppose that $u_0(z) = UzH^{-1}$ in $0 < z < H$. It follows from (2.17*a*, *b*) that

$$\gamma = -\frac{mH(\sigma^2-f^2)}{flU}[\exp\{-\mathrm{i}flU(\sigma^2-f^2)^{-1}\}-1]. \tag{2.20}$$

Over-reflection occurs for $\sigma flU < 0$, and the conditions for resonant over-reflection are

$$\frac{flU}{\sigma^2-f^2} = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots \tag{2.21a}$$

and

$$mH = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots. \tag{2.21b}$$

In figure (2(*b*) we show a contour plot of $|R/I|$ (2.13*a*) as a function of $|\sigma/f|$ and lH. Note that resonant over-reflection now occurs at the single frequency σ_c, where

$$\left(\frac{\sigma_\mathrm{c}^2-f^2}{f^2}\right)^{\frac{1}{2}} = \frac{|U|}{2NH}. \tag{2.22}$$

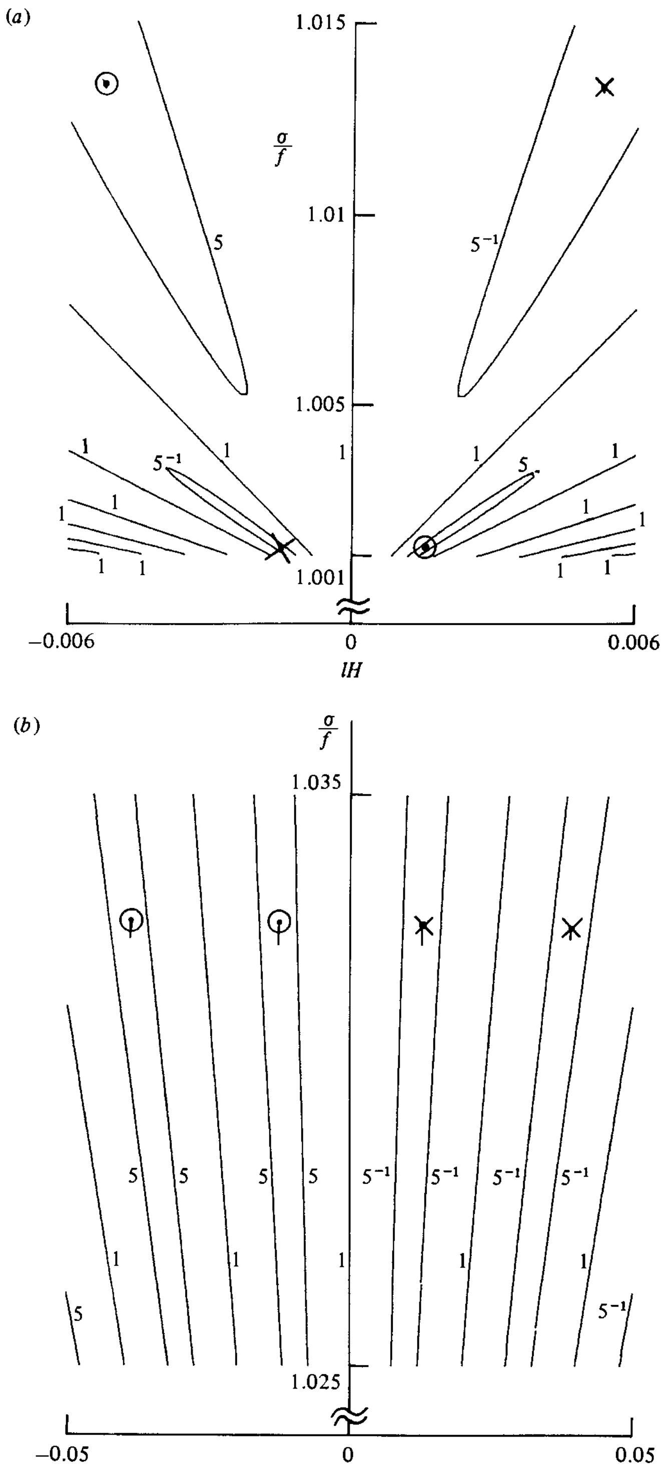


FIGURE 2. A contour plot of the reflection coefficient $|R/I|$ as a function of $|\sigma/f|$ and lH. (*a*) 'Slab' flow, $u_0(z) = U_s$; (*b*) linear shear, $u_0(z) = UzH^{-1}$. In both cases $N = 3.1 \times 10^{-3}\ \mathrm{s}^{-1}$, $f = 10^{-4}\ \mathrm{s}^{-1}$, $H = 50$ m and (*a*) $U_s = 4\ \mathrm{cm\ s}^{-1}$, (*b*) $U = 8\ \mathrm{cm\ s}^{-1}$. Circles (○) and crosses (×) indicate points of resonant over-reflection and absorption respectively.

For $N = 3.1\times10^{-2}\ \mathrm{s^{-1}}$, $|U| = 8\ \mathrm{cm\ s^{-1}}$, $f = 10^{-4}\ \mathrm{s^{-1}}$ and $H = 50$ m, $|\sigma_c f^{-1}| = 1.0327$. Corresponding to this single frequency for resonant over-reflection there is now an increasing set of wavenumbers; this result is quite different from that of case (*a*). Resonant absorption also occurs at the single frequency σ_c at precisely those wavenumbers of opposite sign to the wavenumbers for resonant over-reflection.

(*c*) '*Slab' flow with linear shear*. Finally, suppose that $u_0(z)$ is constant and equal to $U_0(\delta)$ in $\delta \leqslant z \leqslant H$, and that $u_0(z) = U_0 z\delta^{-1}$ in $0 \leqslant z \leqslant \delta$. This case includes the two previous cases; when $\delta \to 0$ we recover case (*a*) where $U_0(0) = U_s$, and when $\delta \to H$ we recover case (*b*) where $U_0(H) = U$. It follows from (2.17*a, b*) that

$$\gamma = \frac{mH}{\psi}\left[\frac{\delta}{H}+\left\{\mathrm{i}\left(1-\frac{\delta}{H}\right)\psi-\frac{\delta}{H}\right\}\exp(-\mathrm{i}\psi)\right], \tag{2.23a}$$

where

$$\psi = \frac{flU_0(\delta)}{\sigma^2-f^2}. \tag{2.23b}$$

The condition $\gamma = 1$ for resonant over-reflection is thus

$$\tan\psi = -\frac{H-\delta}{\delta}\psi \tag{2.24a}$$

and

$$\frac{m\delta}{\psi}\left[1-\frac{\{\delta^2+\psi^2(H-\delta)^2\}^{\frac{1}{2}}}{\delta}\right] = 1. \tag{2.24b}$$

Here the sign of the square root is that of $\cos\psi$ (or equivalently of $-\psi\sin\psi$). When $\delta\to 0$ or H these conditions reduce to (2.19*a, b*) and (2.21*a, b*) respectively. In order to examine the transition between these limiting cases, we first normalize by putting $U_0(\delta) = U_s(1-\delta/2H)^{-1}$, so that for all values of δ the total transport in the mixed layer is HU_s; note that U in case (*b*) is then $2U_s$. In figure 3 we show a plot of the location of the points of resonant over-reflection in the (σ, l)-plane as a function of δ; also shown are the points of resonant absorption which are determined from (2.24*a, b*) with the right-hand side replaced with -1. Note that when $\delta \to 0$ resonant over-reflection can occur for both signs of l (with $\sigma > 0$), but that when $\delta \to H$ resonant over-reflection occurs only for $\sigma l < 0$. Thus when $\sigma l < 0$ the points of resonant over-reflection connect smoothly as δ/H increases from 0 to 1. However, for $\sigma l > 0$ the points of resonant over-reflection move to infinity (with $\sigma^2 l^{-1}$ remaining constant); of course these branches are ultimately outside the range of the model, which requires $\sigma^2 \ll N^2$.

Equation (2.11) and the discussion that follows that equation show that it is the non-conservation of wave action flux within the mixed layer which provides the mechanism for over-reflection. Further, (2.9*a*) shows that it is the 'dissipative' term D (2.9*b*) that allows for divergence of wave action flux. This is in strong contrast to most previous examples of over-reflection for which a critical layer is needed to provide a source of wave action flux (see Acheson 1976; McIntyre & Weissman 1978).

Note that here D is non-zero only because the basic flow $\boldsymbol{u}_0(z)$ is maintained by the stress $\boldsymbol{F}_0(z)$, which in turn is a parametrization of the turbulent Reynolds stress induced by the wind stress $\boldsymbol{\tau}$. In figure 4 (*a*) we show the distribution of the wave-action flux B_3, and the dissipative term D, for various values of δ/H. Note that B_3 must be zero at $z = H$ and $\sigma B_3 < 0$ at $z = 0$, and is constant in $z \leqslant 0$. It is clear that for the examples we consider it is the shear at the base of the mixed layer which is the

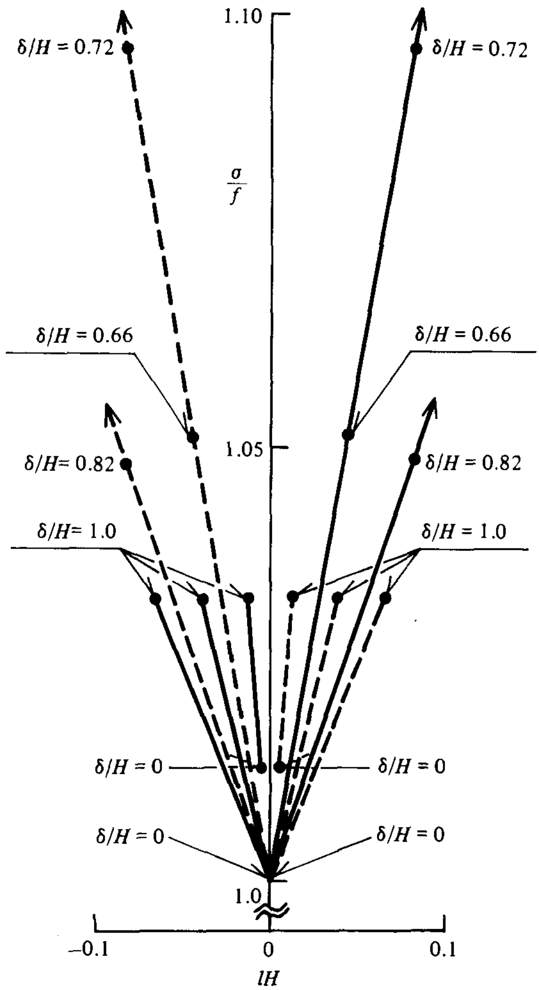


FIGURE 3. A plot of the points of resonant over-reflection (——), and the points of resonant absorption (– – –) as functions of $|\sigma/f|$ and lH for various values of δ, where $u_0(z) = U_0(\delta)$ in $\delta \leqslant z \leqslant H$, and $U_0 z\delta^{-1}$ in $0 \leqslant z \leqslant \delta$ (case c). In all cases $N = 3.1 \times 10^{-3}\ \mathrm{s}^{-1}$, $f = 10^{-4}\ \mathrm{s}^{-1}$, $H = 50$ m and $U_s = 4\ \mathrm{cm\ s}^{-1}$, where $U_0(\delta) = U_s(1-\delta/2H)^{-1}$.

source for the over-reflection. By contrast, we also show in figure 4 (b) the distribution of B_3 and D for regular reflection when $|R/I| = 1$; in this case B_3 must be zero in $z \leqslant 0$.

3. Wave action equation

In this section we propose to discuss the mechanism of over-reflection in a more general setting than that of §2. First suppose that ζ_1 etc. depend smoothly on the ensemble parameter θ such that

$$\zeta_1(t, x, y, z; \theta + 2\pi) = \zeta_1(t, x, y, z; \theta). \tag{3.1}$$

This definition includes, but is more general than, the solutions (2.6a) discussed in §2. We then define the averaging operator

$$\langle \ldots \rangle = \frac{1}{2\pi}\int_0^{2\pi} (\ldots)\, \mathrm{d}\theta. \tag{3.2}$$

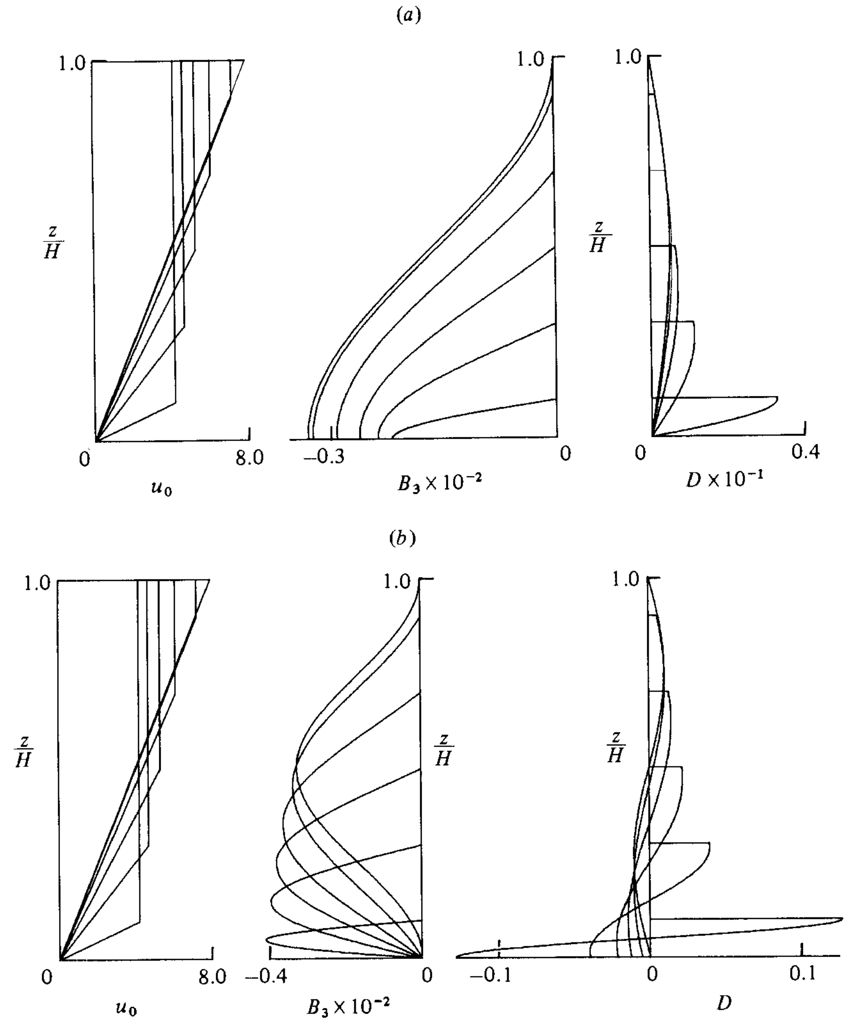


FIGURE 4. A plot of the vertical wave action flux B_3 and the 'dissipative' term D in the mixed layer for various values of δ, where $u_0(z)$ is case (c) (see caption for figure 3). (a) Resonant over-reflection; (b) regular reflection. In both cases only the branch corresponding to the smallest value of $|\psi|$ (2.23b) is shown.

The averaging operator commutes with derivatives such as $\partial/\partial t$ etc., and has other obvious properties (see Andrews & McIntyre 1978a). The wave action equation can now be obtained as a special case of the general theory of Andrews & McIntyre (1978b), or, more directly, by multiplying (2.5a–c) with $\partial\xi_1/\partial\theta$, $\partial\eta_1/\partial\theta$ and $\partial\zeta_1/\partial\theta$ respectively, averaging, and adding the resultant equations. The result is

$$\frac{\mathrm{D}A}{\mathrm{D}t}+\frac{\partial B_1}{\partial x}+\frac{\partial B_2}{\partial y}+\frac{\partial B_3}{\partial z}=D, \tag{3.3a}$$

where

$$A=\rho_0\left\langle\frac{\partial\xi_1}{\partial\theta}\left(\frac{\mathrm{D}\xi_1}{\mathrm{D}t}-\tfrac{1}{2}f\eta_1\right)+\frac{\partial\eta_1}{\partial\theta}\left(\frac{\mathrm{D}\eta_1}{\mathrm{D}t}+\tfrac{1}{2}f\xi_1\right)\right\rangle, \tag{3.3b}$$

$$\boldsymbol{B}=\left\langle p_1\frac{\partial\xi_1}{\partial\theta}\right\rangle, \tag{3.3c}$$

and

$$D = \rho_0\left\langle \zeta_1\left(f\frac{\partial u_0}{\partial z}\frac{\partial \eta_1}{\partial \theta} - f\frac{\partial v_0}{\partial z}\frac{\partial \xi_1}{\partial \theta}\right)\right\rangle. \tag{3.3d}$$

Here $\boldsymbol{B}$ has components (B_1, B_2, B_3), and $\boldsymbol{\xi}_1$ is the Lagrangian particle displacement with components (ξ_1, η_1, ζ_1). A is the wave action density, and $\boldsymbol{B}$ is the wave action flux. D is the 'dissipative' term, and is non-zero only because the basic shear flow is maintained by the stress $\boldsymbol{F}_0$ (see (2.1)). For the special case of §2 when ζ_1 etc. are given by (2.6*a*), it is readily verified that B_3 reduces to (2.8), D is given by (2.9*b*), and the wave action equation (3.3*a*) reduces to (2.9*a*).

As we have shown in §2, it is the presence of the 'dissipative' term D which is responsible for over-reflection. Using (2.1), D (3.3*d*) can be cast in the form

$$D = \left\langle \zeta_1\left(\frac{\partial \xi_1}{\partial \theta}\frac{\partial^2 F_0}{\partial z^2} + \frac{\partial \eta_1}{\partial \theta}\frac{\partial^2 G_0}{\partial z^2}\right)\right\rangle. \tag{3.4}$$

Hence for D to be non-zero, the horizontal particle displacement must have a component in the direction of $\partial^2\boldsymbol{F}_0/\partial z^2$, and this component must be out of phase with the vertical particle displacement.

To obtain an equation describing wave energy and its interaction with the basic shear flow, we multiply (2.5*a, b, c*) with $\mathrm{D}\xi_1/\mathrm{D}t$, $\mathrm{D}\eta_1/\mathrm{D}t$ and $\mathrm{D}\zeta_1/\mathrm{D}t$ respectively, average the result, and add the resultant equations. We obtain

$$\frac{\mathrm{D}E}{\mathrm{D}t} + \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -R_{13}\frac{\partial u_0}{\partial z} - R_{23}\frac{\partial v_0}{\partial z} + D_E, \tag{3.5a}$$

where

$$E = \tfrac{1}{2}\rho_0\left\langle\left(\frac{\mathrm{D}\xi_1}{\mathrm{D}t}\right)^2 + \left(\frac{\mathrm{D}\eta_1}{\mathrm{D}t}\right)^2 + N^2\zeta_1^2\right\rangle, \tag{3.5b}$$

$$\boldsymbol{F} = \left\langle p_1\frac{\mathrm{D}\boldsymbol{\xi}_1}{\mathrm{D}t}\right\rangle, \tag{3.5c}$$

$$R_{13} = -\left\langle p_1\frac{\partial \zeta_1}{\partial x}\right\rangle, \quad R_{23} = -\left\langle p_1\frac{\partial \zeta_1}{\partial y}\right\rangle \tag{3.5d}$$

and

$$D_E = \rho_0\left\langle \zeta_1\left(f\frac{\partial u_0}{\partial z}\frac{\mathrm{D}\eta_1}{\mathrm{D}t} - f\frac{\partial v_0}{\partial z}\frac{\mathrm{D}\xi_1}{\mathrm{D}t}\right)\right\rangle. \tag{3.5e}$$

Here E is the wave energy density and $\boldsymbol{F}$ is the wave energy flux. R_{13} and R_{23} are components of the radiation stress tensor (see Andrews & McIntyre 1978*a*), while D_E is the 'dissipative' term. For the special case of §2 when ζ_1 etc. are given by (2.6*a*), it may be shown that $F_3 = \omega B_3$, $D_E = \omega D$ and $R_{13} = kB_3$, $R_{23} = lB_3$. The wave energy equation (3.5*a*) is readily seen to be equivalent to the wave action equation (3.3*a*) in this special case, but is clearly less instructive as wave energy is exchanged with the basic flow both through the radiation stress terms and the 'dissipative' term.

4. Wave-induced mean flow

In this section we shall complement the discussion of §3 by calculating the wave-induced mean flow. In the Boussinesq and hydrostatic approximations, the Eulerian equations of motion are

$$\frac{\mathrm{d}u}{\mathrm{d}t} - fv + \frac{1}{\rho_0}\frac{\partial p}{\partial x'} = \frac{1}{\rho_0}\frac{\partial F_0}{\partial z'}, \quad \frac{\mathrm{d}v}{\mathrm{d}t} + fu + \frac{1}{\rho_0}\frac{\partial p}{\partial y'} = \frac{1}{\rho_0}\frac{\partial G_0}{\partial z'}, \tag{4.1a, b}$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z'} + r = 0, \quad \frac{\mathrm{d}r}{\mathrm{d}t} - N^2(z')w = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.1c–e}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x'} + v\frac{\partial}{\partial y'} + w\frac{\partial}{\partial z'}. \tag{4.1f}$$

Here $\boldsymbol{x}'$ with components (x', y', z') is the Eulerian coordinate such that the fluid particle at $\boldsymbol{x}'$ has velocity $\boldsymbol{u}$; r is the buoyancy $g(\rho-\rho_0)/\rho_0$. The wave-induced mean flow could now be calculated by averaging these equations, and then calculating the Reynolds stresses and buoyancy fluxes to the second order in wave amplitude from the results of §2.

However, it is more revealing to adapt the generalized Lagrangian-mean formulation of Andrews & McIntyre (1978*a*). Thus let $\boldsymbol{x}$ be a generalized Lagrangian coordinate, and let $\boldsymbol{\xi}(t, \boldsymbol{x})$ be the particle displacement defined so that

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\xi}. \tag{4.2}$$

Then, for any given velocity field $\boldsymbol{u}$ there is a unique Lagrangian-mean velocity $\bar{\boldsymbol{u}}^{\mathrm{L}}$ such that when the point $\boldsymbol{x}$ moves with velocity $\bar{\boldsymbol{u}}^{\mathrm{L}}$ the point $\boldsymbol{x}'$ moves with velocity $\boldsymbol{u}$, and such that

$$\langle \xi \rangle = 0. \tag{4.3}$$

Here we recall that the averaging operation has been defined by (3.2). With these definitions the material time derivative (4.1*f*) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \bar{u}^{\mathrm{L}}\frac{\partial}{\partial x} + \bar{v}^{\mathrm{L}}\frac{\partial}{\partial y} + \bar{w}^{\mathrm{L}}\frac{\partial}{\partial z}. \tag{4.4}$$

Since the Eulerian flow is incompressible (see (4.1*e*)), it follows that

$$\frac{1}{J}\frac{\mathrm{d}J}{\mathrm{d}t} + \frac{\partial \bar{u}^{\mathrm{L}}}{\partial x} + \frac{\partial \bar{v}^{\mathrm{L}}}{\partial y} + \frac{\partial \bar{w}^{\mathrm{L}}}{\partial z} = 0, \tag{4.5a}$$

where

$$J = \det\left[\partial \boldsymbol{x}'/\partial \boldsymbol{x}\right] \tag{4.5b}$$

is the Jacobian of the mapping from $\boldsymbol{x}$ to $\boldsymbol{x}'$, and is itself a mean quantity. Equation (4.5*a*) shows that the Lagrangian-mean velocity is generally divergent. Next we define K_{ij} as the (i,j)-cofactor of J, and so

$$K_{ik}\frac{\partial x_j'}{\partial x_k} = J\delta_{ij} = K_{ki}\frac{\partial x_k'}{\partial x_j}. \tag{4.6}$$

It may then be shown that the momentum equations (4.1*a*–*c*) become

$$\frac{\mathrm{d}u}{\mathrm{d}t} - fv + \frac{1}{\rho_0 J}\frac{\partial}{\partial x_j}(pK_{1j}) = -fv_0(z'), \tag{4.7a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + fu + \frac{1}{\rho_0 J}\frac{\partial}{\partial x_j}(pK_{2j}) = fu_0(z'), \tag{4.7b}$$

$$\frac{1}{\rho_0 J}\frac{\partial}{\partial x_j}(pK_{3j}) + r = 0. \tag{4.7c}$$

Together with (4.1*d*) and (4.5*a*) these form the Lagrangian equations. Note that

$$\boldsymbol{u} = \bar{\boldsymbol{u}}^{\mathrm{L}} + \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}. \tag{4.8}$$

The Lagrangian-mean equations are now obtained by averaging these equations. The result for (4.7*a*, *b*) is

$$\frac{\mathrm{d}\bar{u}^{\mathrm{L}}}{\mathrm{d}t}-f\bar{v}^{\mathrm{L}}+\frac{1}{\rho_0 J}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial x} = -\frac{1}{\rho_0 J}\frac{\partial}{\partial x_j}R_{1j}-\langle f v_0(z')\rangle, \tag{4.9a}$$

$$\frac{\mathrm{d}\bar{v}^{\mathrm{L}}}{\mathrm{d}t}+f\bar{u}^{\mathrm{L}}+\frac{1}{\rho_0 J}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial z} = -\frac{1}{\rho_0 J}\frac{\partial}{\partial x_j}R_{2j}+\langle f u_0(z')\rangle, \tag{4.9b}$$

where

$$R_{ij} = \delta_{ij}\bar{p}^{\mathrm{L}}(J-1)-\left\langle p\frac{\partial \xi_k}{\partial x_i}K_{kj}\right\rangle \tag{4.9c}$$

is the radiation stress tensor. Note that the last terms in (4.9*a*, *b*) are 'dissipative' terms, which appear explicitly in the Lagrangian-mean formulation, but would not appear explicitly in the Eulerian-mean formulation. The remaining three equations are (4.5*a*) and the average of (4.1*d*) and (4.7*c*); these latter two will not be needed in the sequel and serve only to determine the mean pressure and buoyancy.

If a is a measure of wave amplitude, we then write

$$\boldsymbol{u} = \boldsymbol{u}_0(z)+\frac{\mathrm{D}\xi_1}{\mathrm{D}t}+O(a^2) \quad \text{etc.} \tag{4.10a}$$

Then, to $O(a)$, it may be verified that the linearized equations (2.5*a*–*d*) can be obtained from (4.7*a*–*c*). Next let

$$\bar{\boldsymbol{u}}^{\mathrm{L}} = \boldsymbol{u}_0(z)+\bar{\boldsymbol{u}}_2^{\mathrm{L}}+O(a^3) \quad \text{etc.} \tag{4.10b}$$

Then (4.9*a*, *b*) become, to $O(a^2)$,

$$\frac{\mathrm{D}\bar{u}_2^{\mathrm{L}}}{\mathrm{D}t}+\bar{w}_2^{\mathrm{L}}\frac{\partial u_0}{\partial z}-f\bar{v}_2^{\mathrm{L}}+\frac{1}{\rho_0}\frac{\partial \bar{p}_2^{\mathrm{L}}}{\partial x} = -\frac{1}{\rho_0}\frac{\partial}{\partial x_j}R_{1j}-f\frac{\partial^2 v_0}{\partial z^2}\langle\tfrac{1}{2}\zeta_1^2\rangle, \tag{4.11a}$$

$$\frac{\mathrm{D}\bar{v}_2^{\mathrm{L}}}{\mathrm{D}t}+\bar{w}_2^{\mathrm{L}}\frac{\partial v_0}{\partial z}+f\bar{u}_2^{\mathrm{L}}+\frac{1}{\rho_0}\frac{\partial \bar{p}_2^{\mathrm{L}}}{\partial y} = -\frac{1}{\rho_0}\frac{\partial}{\partial x_j}R_{2j}+f\frac{\partial^2 u_0}{\partial z^2}\langle\tfrac{1}{2}\zeta_1^2\rangle, \tag{4.11b}$$

where

$$R_{ij} = -\left\langle p_1\frac{\partial \xi_{1j}}{\partial x_i}\right\rangle. \tag{4.11c}$$

Note that the radiation stress tensor R_{ij} defined by (4.11*c*) agrees with that obtained in the wave energy equation (see (3.5*d*)). The total energy equation is obtained by multiplying (4.11*a*, *b*) by u_0 and v_0 respectively, multiplying (2.1) by $\bar{u}_2^{\mathrm{L}}$, and adding the result to the wave energy equation (3.5*a*):

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\{\rho_0\boldsymbol{u}_0\cdot\bar{\boldsymbol{u}}_2^{\mathrm{L}}+E+(J-1)(\tfrac{1}{2}\rho_0 u_0^2+\tfrac{1}{2}\rho_0 v_0^2)\}+\boldsymbol{\nabla}\cdot\{\boldsymbol{u}_0\bar{p}_2^{\mathrm{L}}+\bar{\boldsymbol{u}}_2^{\mathrm{L}}(\tfrac{1}{2}\rho_0 u_0^2+\tfrac{1}{2}\rho_0 v_0^2)+\boldsymbol{F}\}\\ +\frac{\partial}{\partial x_j}\{u_0R_{ij}+v_0R_{2j}\} = D_E-\rho_0 f v_0\bar{u}_2^{\mathrm{L}}+\rho_0 f u_0\bar{v}_2^{\mathrm{L}}+\rho_0 f\left(v_0\frac{\partial^2 u_0}{\partial z^2}-u_0\frac{\partial^2 v_0}{\partial z^2}\right)\langle\tfrac{1}{2}\zeta_1^2\rangle.\end{aligned} \tag{4.12}$$

Note here that to $O(a^2)$

$$J-1 = -\frac{1}{2}\frac{\partial^2}{\partial x_j\,\partial x_k}\langle\xi_j\,\xi_k\rangle. \tag{4.13}$$

The mean-flow equations (4.11*a*, *b*) and the energy equation (4.12) together demonstrate how the wave-induced mean flow combines with the wave energy to provide the mechanism for over-reflection. Note in particular that for over-reflection the total vertical energy flux must be directed downwards, and the source for this flux is the 'dissipative' term on the right-hand side of (4.12).

In order to illustrate further the role of these equations, we next consider the special case when all mean quantities depend only on t and z; this is more general than the case discussed in §2 when the wave quantities are given by (2.6a) and mean quantities depend only on z. For this special case (4.5a) reduces to

$$\bar{w}_2^{\mathrm{L}} = \frac{\partial^2}{\partial t\,\partial z}\langle\tfrac{1}{2}\zeta_1^2\rangle. \tag{4.14}$$

Here we have applied the boundary condition that $\bar{w}_2^{\mathrm{L}}$ must vanish at $z = H$. Next we observe that $R_{1j} = kB_j$ and $R_{2j} = lB_j$, where we are assuming that the x- and y-dependences of the wave-like quantities are given by expressions of the form (2.6a). Thus the radiation stress terms in (4.11a, b) can be eliminated using the wave action equation (3.3a). We find that

$$\frac{\partial}{\partial t}\left\{\rho_0\left(\bar{u}_2^{\mathrm{L}} + \frac{\partial u_0}{\partial z}\frac{\partial}{\partial z}\langle\tfrac{1}{2}\zeta_1^2\rangle\right) - kA\right\} - \rho_0 f\bar{v}_2^{\mathrm{L}} = -kD - \rho_0 f\frac{\partial^2 v_0}{\partial z^2}\langle\tfrac{1}{2}\zeta_1^2\rangle, \tag{4.15a}$$

$$\frac{\partial}{\partial t}\left\{\rho_0\left(\bar{v}_2^{\mathrm{L}} + \frac{\partial v_0}{\partial z}\frac{\partial}{\partial z}\langle\tfrac{1}{2}\zeta_1^2\rangle\right) - lA\right\} + \rho_0 f\bar{u}_2^{\mathrm{L}} = -lD + \rho_0 f\frac{\partial^2 u_0}{\partial z^2}\langle\tfrac{1}{2}\zeta_1^2\rangle. \tag{4.15b}$$

These equations show that for steady waves the wave-induced mean flow $(\bar{u}_2^{\mathrm{L}}, \bar{v}_2^{\mathrm{L}})$ is non-zero only because of the 'dissipative' terms on the right-hand side of (4.15a, b). This result is analogous to the Charney–Drazin theorems in stratospheric meteorology. The total energy equation becomes

$$\frac{\partial}{\partial t}\left\{\rho_0\,\boldsymbol{u}_0\cdot\bar{\boldsymbol{u}}_2^{\mathrm{L}} + E + \frac{\partial}{\partial z}\langle\tfrac{1}{2}\zeta_1^2\rangle\frac{\partial}{\partial z}(\tfrac{1}{2}\rho_0\,u_0^2 + \tfrac{1}{2}\rho_0\,v_0^2)\right\} + \frac{\partial}{\partial z}\{F_3 + (ku_0 + lv_0)\,B_3\} = \ldots. \tag{4.16}$$

Here the right-hand side is the same as that in (4.12). For steady waves

$$F_3 + (ku_0 + lv_0)\,B_3 = \sigma B_3, \tag{4.17}$$

and the right-hand side of (4.16) is σD. The total energy equation in this instance reduces to the wave action equation, and again emphasizes the role that D plays in the mechanism of over-reflection.

5. Linear stability theory

Up to this point we have assumed that the perturbations to the basic flow consist only of neutral waves. In this section we shall examine the stability of the basic flow by searching for solutions of the linearized equations (2.5a–d), which have the form (2.6a, b), where the wavenumber (k, l) is real but the frequency $\sigma = \sigma_{\mathrm{r}} + i\sigma_{\mathrm{i}}$ is complex. For instability we require that $\sigma_{\mathrm{i}} > 0$. In $z < 0$ the solution is given by (2.10a) with $I = 0$, and $m = m_{\mathrm{r}} + im_{\mathrm{i}}$ given by (2.10b), with $m_{\mathrm{i}} > 0$. The calculations of §2 which led to (2.13a, b) remain valid when $\sigma_{\mathrm{i}} \neq 0$, and thus, since $I = 0$ here, instability is determined by the condition

$$\gamma = 1, \tag{5.1}$$

where γ is given by (2.13b). Since, when $\sigma_{\mathrm{i}} = 0$, this is precisely the condition for resonant over-reflection, we can anticipate that there may be instability for those points in the (σ_{r}, l)-plane that are close to points of resonant over-reflection. In a similar way, damped waves with $\sigma_i < 0$ can be sought with the solution given by (2.10*a*) with $R = 0$ and $m_i < 0$; these damped waves are therefore given by the condition $\gamma = -1$ (see (2.13*a*) with $R = 0$), and we can anticipate that they will exist in the (σ_r, l)-plane close to points of resonant absorption.

For small σ_i (i.e. close to points of resonant over-reflection or resonant absorption)

$$m_i \approx \sigma_i V^{-1}, \tag{5.2}$$

where $V = \partial\sigma/\partial m$ evaluated at $\sigma_i = 0$ is the vertical group velocity. Since our sign convention requires V to be positive (see (2.11) and the following discussion), it follows that σ_i and m_i have the same sign in accord with the discussion above. Also the trapping scale for unstable or damped waves is $O(V\sigma_i^{-1})$ (for a more extensive discussion of this aspect see McIntyre & Weissman 1978).

If (σ_0, l_0) is a point in the (σ, l)-plane for resonant over-reflection, then for a point (σ, l) we may solve (5.1) approximately when $\delta\sigma$ and δl are small, where

$$\left.\begin{aligned} \delta\sigma &= \sigma - \sigma_0 = \delta\sigma_r + \mathrm{i}\,\delta\sigma_i, \\ \delta l &= l - l_0. \end{aligned}\right\} \tag{5.3}$$

For the two special cases (*a*) and (*b*) of §2 the results are as follows.

(*a*) '*Slab*' *flow* ($u_0(z) = U_s$ in $0 < z < H$):

$$\delta\sigma_r = \frac{(\sigma_0^2 - f^2)(1 + 2\psi_s^2)}{\sigma_0 l_0 (1 + 4\psi_s^2)}\delta l, \tag{5.4a}$$

$$\delta\sigma_i = \frac{(\sigma_0^2 - f^2)\psi_s}{\sigma_0 l_0 (1 + 4\psi_s^2)}\delta l, \tag{5.4b}$$

where

$$\psi_s = \frac{f l_0 U_s}{\sigma_0^2 - f^2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots. \tag{5.4c}$$

(*b*) *Linear shear flow* ($u_0(z) = UzH^{-1}$ in $0 < z < H$):

$$\delta\sigma_r = \frac{(\sigma_0^2 - f^2)\psi_0^2}{2\sigma_0 l_0 (1 + \psi_0^2)}\delta l, \quad \delta\sigma_i = \frac{(\sigma_0^2 - f_0^2)\psi_0}{2\sigma_0 l_0 (1 + \psi_0^2)}\delta l, \tag{5.5a, b}$$

where

$$\psi_0 = \frac{f l_0 U}{\sigma_0^2 - f^2} = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots. \tag{5.5c}$$

These results, in particular (5.4*b*) and (5.5*b*), show that points (σ_0, l_0) for resonant over-reflection are the endpoints of a curve in the (σ, l)-plane representing unstable waves. Since $\delta\sigma_i > 0$ for instability, the unstable branch is one-sided in a neighbourhood of (σ_0, l_0), and must terminate there. A similar analysis for damped waves shows that these emanate from points for resonant absorption.

The continuation of the curves in the (σ, l)-plane for the unstable and damped waves can be determined numerically from (5.1), and the results for the special cases (*a*) and (*b*) of §2 are shown in figures 5(*a*, *b*) respectively. In both cases (*a*) and (*b*), the growth rates are very small (for instance, in case (*a*) the maximum growth rate $\sigma_i \approx 0.001\,|f|$ when $N = 3.1 \times 10^{-3}\,\mathrm{s}^{-1}$, $|U_s| = 4.0\,\mathrm{cm\,s^{-1}}$ and $H = 50$ m), and the maximum growth rate occurs for smaller values of $|lH|$ than the corresponding value $|l_0 H|$ for resonant over-reflection. In both cases (*a*) and (*b*) the maximum growth rate occurs for that branch with the smallest value of $|\psi_s|$ or $|\psi_0|$ respectively.

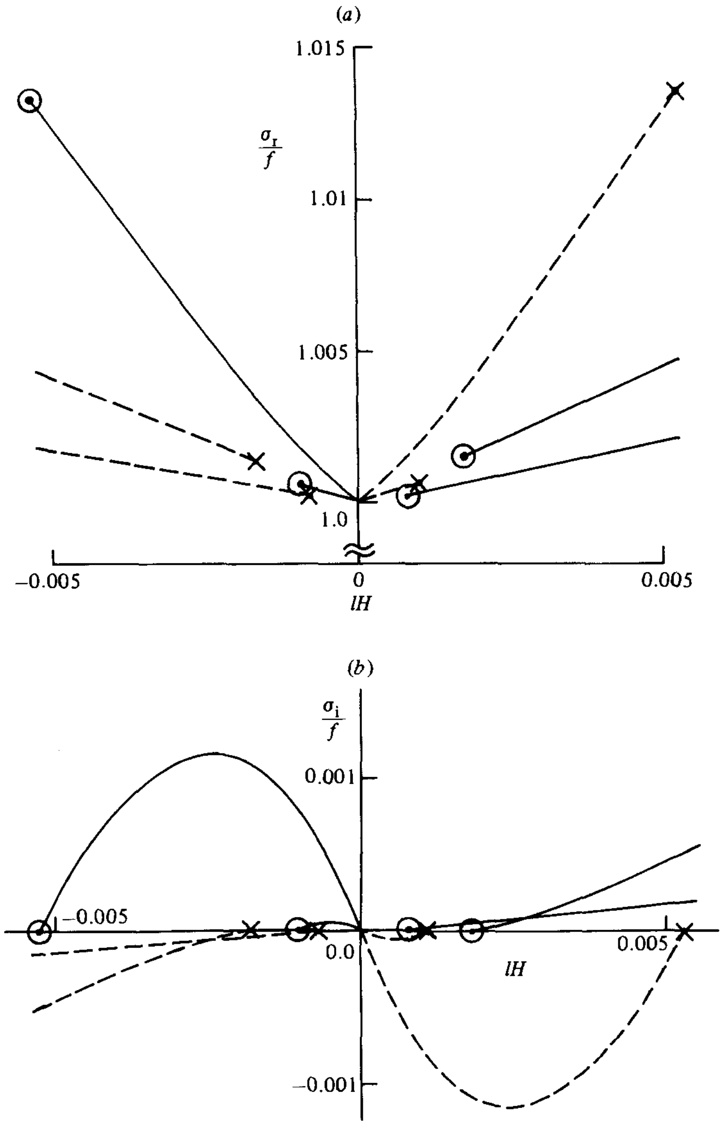
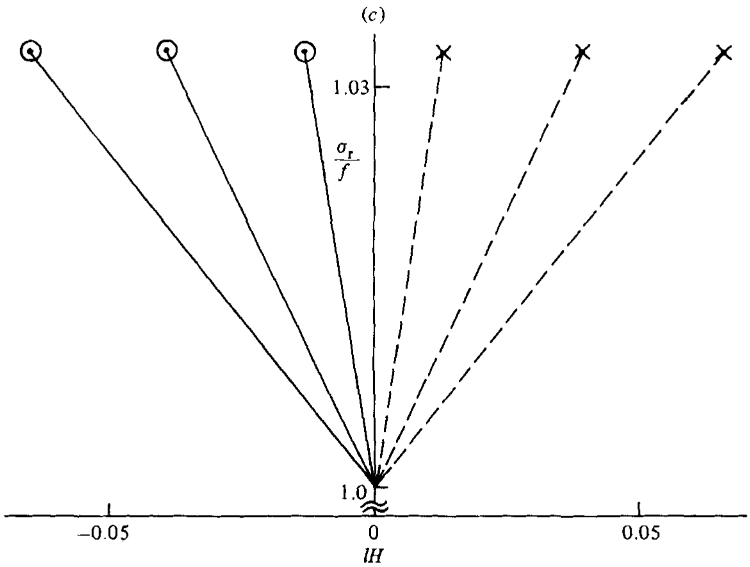


FIGURE 5 (a, b). For caption see facing page.

Since these unstable waves have small growth rates they will also have large trapping scales and will extend a considerable distance into the deep ocean. In practice they may be indistinguishable from the untrapped waves generated by the over-reflection mechanism at the base of the mixed layer.

Finally the unstable waves considered here should not be confused with the unstable waves that would occur when an over-reflected wave is reflected off bottom topography, and propagates upwards, only to be over-reflected again from the base of the mixed layer. Successive repetition of this process leads to temporally growing waves (for a discussion of this process for internal waves in the atmosphere see Lindzen & Rosenthal 1976). Mollo-Christensen (1977) has analysed one aspect of this problem when the base of the mixed layer is modelled as a vortex sheet, and suggested that near-inertial waves may be unstable. The dispersion relation for this second class



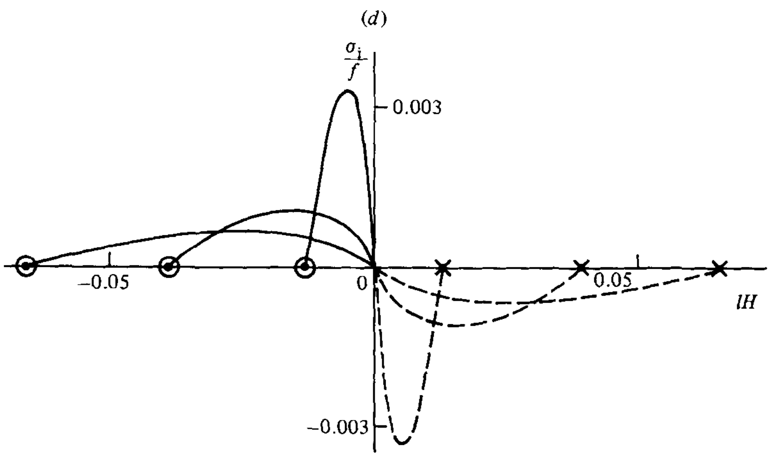


FIGURE 5. A plot of the points representing unstable waves (——) and points representing damped waves (– – –) as functions of σ_r/f or σ_i/f, and lH. Circles (○) and crosses (×) indicate the points of resonant over-reflection and absorption respectively for neutral waves. (a) σ_r/f for 'slab' flow, $u_0(z) = U_s$; (b) σ_i/f for 'slab' flow; (c) σ_r/f for linear shear flow, $u_0(z) = UzH^{-1}$; (d) σ_i/f for linear shear flow.

of unstable waves can be determined from the work of §2 by imposing a rigid boundary at $z = -h$ and requiring that $\hat{\zeta}_1$ given by (2.10a) should vanish there. Thus

$$\frac{R}{I} = -\exp(-2imh), \tag{5.6}$$

and substituting this into (2.13a) leads to the dispersion relation

$$i\gamma = \tanh mh. \tag{5.7}$$

We shall not attempt to solve this dispersion relation here. However, for the oceanic case when $h \gg H$, we anticipate that the growth rates associated with this second class of unstable waves will be $O(H/h)$ relative to the growth rates for the first class of unstable waves shown in figures 5(*a*, *b*).

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